National Exams

December 2009

07-Elec-B1, Digital Signal Processing

3 hours duration

NOTES:

1. If doubt exists as to the interpretation of any question, the candidate is urged to submit with the answer paper, a clear statement of any assumptions made.
2. This is a CLOSED BOOK EXAM. A Casio or Sharp approved calculator is permitted.
3. FOUR(4) questions constitute a complete paper. The first four questions as they appear in the answer book will be marked.
4. All questions are of equal value.
5. Clarity and organization of the answer are important.
6. LIST OF IMPORTANT EQUATIONS CAN BE FOUND ON LAST PAGE

1. (**25 marks total**) Consider the rectangular-pulse, continuous-time signal $x(t) = rect(t)$. Take eight samples of $x(t)$ at a rate of five samples per second, with the first sample at $t = -0.6$. No signal truncation occurs when these samples are used. Draw the signal flow graph for an eight-point FFT, labeling all inputs, outputs, and multipliers. Use this signal flow graph to compute the spectrum samples $X(m)$ for the sampled signal.

2. (**25 marks total**) Consider the cascade of three causal first-order LTI discrete-time systems shown in Figure below (Figure 1), where

$$H_1(z) = \frac{2 - 0.3z^{-1}}{1 + 0.5z^{-1}}, H_2(z) = \frac{0.4 + z^{-1}}{1 + 0.4z^{-1}}, H_3(z) = \frac{3}{1 + 0.5z^{-1}} \tag{1}$$

(a) Determine the transfer function of the overall system as a ratio of two polynomials in z^{-1}.

(b) Determine the difference equation characterizing the overall system.

(c) Develop the realization of the overall system with each section realized in direct form II.

(d) Develop a parallel form I realization of the overall system.

(e) Determine the impulse response of the overall system in closed form.

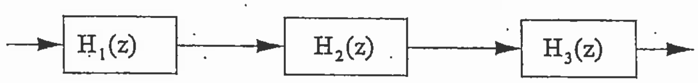


Figure 1:

3. **(25 marks total)** A linear time-invariant system with system function:

$$H(z) = \frac{0.2(1+z^{-1})^6}{(1-2z^{-1}+\frac{7}{8}z^{-2})(1+z^{-1}+\frac{1}{2}z^{-2})(1-\frac{1}{2}z^{-1}+z^{-2})} \tag{2}$$

is to be implemented using a flow graph of the form shown in below Figure 2.

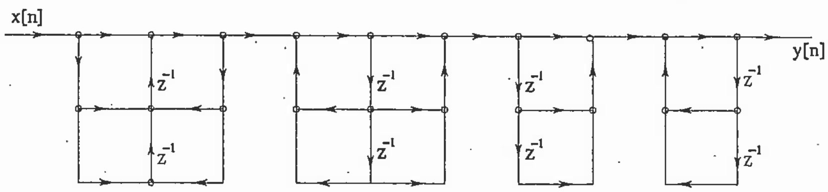


Figure 2:

(a) Fill in all the coefficients in the diagram of Figure 2. Is your solution unique?

(b) Define appropriate node variables in Figure 2., and write the set of difference equations that is represented by the flow graph.

4. **(25 marks total)** Consider the system described by the difference equation:

$$y[n] = \frac{1}{2}y[n-1] + x[n] + \frac{1}{2}x[n-1]. \qquad (3)$$

(a) Determine the impulse (unit sample) response sequence $h[n]$.

(b) Determine the frequency response function $H(e^{j\theta})$.

i. from the unit sample response;

ii. from the difference equation.

(c) Determine the response of the system to the input $x[n] = \cos(\frac{\pi}{2}n + \frac{\pi}{4})$.

5. **(25 marks total)** Consider the system shown in Figure 3. The input to this system is the bandlimited signal whose Fourier transform is shown in Figure 4 with $\Omega_0 = \frac{\pi}{T}$. The discrete-time LTI system in Figure 3. has the frequency response shown in Figure 5.

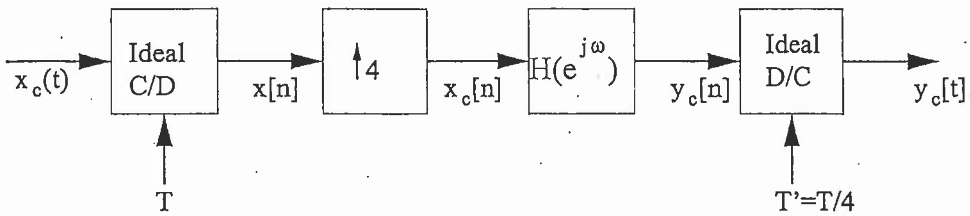


Figure 3:

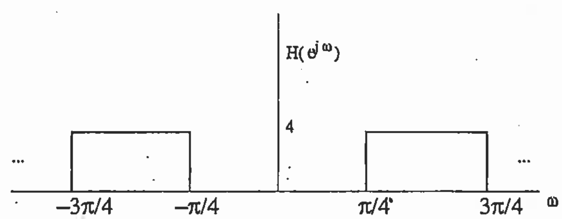


Figure 4:

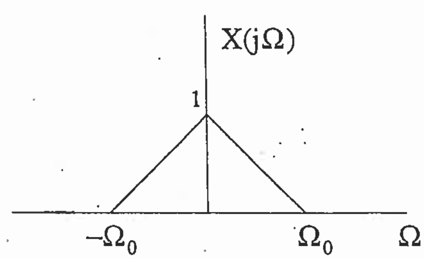


Figure 5:

(a) Sketch the Fourier transforms $X(e^{j\omega}), X_e(e^{j\omega}), Y_e(e^{j\omega})$, and $Y_c(j\Omega)$.

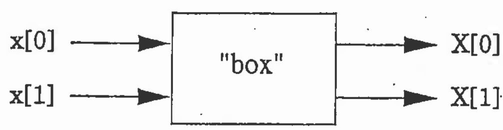
(b) For the general case when $X_c(j\Omega) = 0$ for $|\Omega| \geq \pi/T$, express $Y_c(j\Omega)$ in terms of $X_c(j\Omega)$. Also, give a general expression for $y_c(t)$ in terms of $x_c(t)$ when $x_c(t)$ is band-limited in this manner.

6. **(25 marks total)** Let $u[n] = \{a, b, c\}$ and $v[n] = \{0, 1\}$ be two finite-length, real-valued sequences.

   (a) (3pts) Determine $z_l[n] = u[n] \circledast v[n]$, the linear convolution of the sequences $u[n]$ and $v[n]$.

   (b) (3pts) Determine $z_c[n] = u[n] \text{③} v[n]$, the 3-point circular convolution of the sequences $u[n]$ and $v[n]$.

   (c) (5pts) Let $\{x[n]\} = \{x[0], x[1]\}$ be a 2-element sequence and let $\{X[k]\} = \{X[0], X[1]\}$ be the 2-point Discrete Fourier Transform (DFT) of $\{x[n]\}$. Determine the signal flow diagram for a 2-input, 2-output system, the "box", where $\{x[n]\}$ is the input and $\{X[k]\}$ is the output.



   (d) (4pts) Assume that you are ONLY allowed to perform scalar multiplications external to the "box" you determined in part (c). Show how you would use the "box" such that its output is $x[n]$ when $\{X[k]\}$ is the input.

   (e) (5pts) Determine $z_l[n]$ using 2-point DFT and 2-point IDFT operations. (You may want to use what you derived in parts (c) and (d).)

   (f) (5pts) Let $v[n]$ be a length L_v sequence, and let $u[n]$ be a right-hand sequence (a sequence that is non-zero for $n \leq 0$). You are asked to calculate $z_l[n] = u[n] \circledast v[n]$ using "block filtering" (i.e., using either the overlap-add or the overlap-save method) implemented via L-point DFT/IDFT operations. Determine the minimum allowed value of L.

Trigonometric Identities:

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$$
$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$

$$\sin\alpha\,\sin\beta = \tfrac{1}{2}\cos(\alpha-\beta) - \tfrac{1}{2}\cos(\alpha+\beta)$$
$$\cos\alpha\,\cos\beta = \tfrac{1}{2}\cos(\alpha+\beta) + \tfrac{1}{2}\cos(\alpha-\beta)$$
$$\sin\alpha\,\cos\beta = \tfrac{1}{2}\sin(\alpha+\beta) + \tfrac{1}{2}\sin(\alpha-\beta)$$

Euler's Identity:

$$e^{j\theta} = \cos\theta \,+\, j\,\sin\theta$$

Discrete Fourier Transform:

The N-point DFT of a N-sample sequence $s[n]$ is defined as:

$$S[k] = \sum_{n=0}^{N-1} s[n]\, W_N^{kn}, \qquad k = 0, 1, \ldots, N-1.$$

The sequence $s[n]$ can be recovered from its DFT coefficients using the N-point IDFT:

$$s[n] = \frac{1}{N}\sum_{k=0}^{N-1} S[k]\, W_N^{-kn}, \qquad n = 0, 1, \ldots, N-1.$$

where $W_N = e^{-j2\pi/N}$. The DFT/IDFT relations can also be expressed in matrix form

$$\begin{bmatrix} S[0] \\ \vdots \\ S[N-1] \end{bmatrix} = \mathbf{W}_N \begin{bmatrix} s[0] \\ \vdots \\ s[N-1] \end{bmatrix} \qquad \begin{bmatrix} s[0] \\ \vdots \\ s[N-1] \end{bmatrix} = \mathbf{W}_N^{-1} \begin{bmatrix} S[0] \\ \vdots \\ S[N-1] \end{bmatrix}$$

where the transformation matrices for $N = 2, 3, 4$ are

$$\mathbf{W}_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \qquad \mathbf{W}_2^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$\mathbf{W}_3 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -\frac{1}{2} - \frac{\sqrt{3}}{2}j & -\frac{1}{2} + \frac{\sqrt{3}}{2}j \\ 1 & -\frac{1}{2} + \frac{\sqrt{3}}{2}j & -\frac{1}{2} - \frac{\sqrt{3}}{2}j \end{bmatrix} \qquad \mathbf{W}_3^{-1} = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & -\frac{1}{2} + \frac{\sqrt{3}}{2}j & -\frac{1}{2} - \frac{\sqrt{3}}{2}j \\ 1 & -\frac{1}{2} - \frac{\sqrt{3}}{2}j & -\frac{1}{2} + \frac{\sqrt{3}}{2}j \end{bmatrix}$$

$$\mathbf{W}_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \qquad \mathbf{W}_4^{-1} = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & +j \end{bmatrix}$$

TABLE 9.1 A Short Table of Discrete-Time Fourier Transforms

No.	$x[n]$	$X(\Omega)$	
1	$\delta[n-k]$	$e^{-jk\Omega}$	Integer k
2	$\gamma^n u[n]$	$\frac{e^{j\Omega}}{e^{j\Omega}-\gamma}$	$\|\gamma\| < 1$
3	$-\gamma^n u[-(n+1)]$	$\frac{e^{j\Omega}}{e^{j\Omega}-\gamma}$	$\|\gamma\| > 1$
4	$\gamma^{\|n\|}$	$\frac{1-\gamma^2}{1-2\gamma\cos\Omega+\gamma^2}$	$\|\gamma\| < 1$
5	$n\gamma^n u[n]$	$\frac{\gamma e^{j\Omega}}{(e^{j\Omega}-\gamma)^2}$	$\|\gamma\| < 1$
6	$\gamma^n \cos(\Omega_0 n+\theta)u[n]$	$\frac{e^{j\Omega}[e^{j\Omega}\cos\theta-\gamma\cos(\Omega_0-\theta)]}{e^{j2\Omega}-(2\gamma\cos\Omega_0)e^{j\Omega}+\gamma^2}$	$\|\gamma\| < 1$
7	$u[n]-u[n-M]$	$\frac{\sin(M\Omega/2)}{\sin(\Omega/2)}e^{-j\Omega(M-1)/2}$	
8	$\frac{\Omega_c}{\pi}\operatorname{sinc}(\Omega_c n)$	$\sum_{k=-\infty}^{\infty}\operatorname{rect}\left(\frac{\Omega-2\pi k}{2\Omega_c}\right)$	$\Omega_c \le \pi$
9	$\frac{\Omega_c}{2\pi}\operatorname{sinc}^2\left(\frac{\Omega_c n}{2}\right)$	$\sum_{k=-\infty}^{\infty}\Delta\left(\frac{\Omega-2\pi k}{2\Omega_c}\right)$	$\Omega_c \le \pi$
10	$u[n]$	$\frac{e^{j\Omega}}{e^{j\Omega}-1}+\pi\sum_{k=-\infty}^{\infty}\delta(\Omega-2\pi k)$	
11	1 for all n	$2\pi\sum_{k=-\infty}^{\infty}\delta(\Omega-2\pi k)$	
12	$e^{j\Omega_0 n}$	$2\pi\sum_{k=-\infty}^{\infty}\delta(\Omega-\Omega_0-2\pi k)$	
13	$\cos\Omega_0 n$	$\pi\sum_{k=-\infty}^{\infty}\delta(\Omega-\Omega_0-2\pi k)+\delta(\Omega+\Omega_0-2\pi k)$	
14	$\sin\Omega_0 n$	$j\pi\sum_{k=-\infty}^{\infty}\delta(\Omega+\Omega_0-2\pi k)-\delta(\Omega-\Omega_0-2\pi k)$	
15	$(\cos\Omega_0 n)u[n]$	$\frac{e^{j2\Omega}-e^{j\Omega}\cos\Omega_0}{e^{j2\Omega}-2e^{j\Omega}\cos\Omega_0+1}+\frac{\pi}{2}\sum_{k=-\infty}^{\infty}\delta(\Omega-2\pi k-\Omega_0)+\delta(\Omega-2\pi k+\Omega_0)$	

TABLE 9.2 Properties of the DTFT

Operation	$x[n]$	$X(\Omega)$
Linearity	$a_1x_1[n]+a_2x_2[n]$	$a_1X_1(\Omega)+a_2X_2(\Omega)$
Conjugation	$x^*[n]$	$X^*(-\Omega)$
Scalar multiplication	$ax[n]$	$aX(\Omega)$
Multiplication by n	$nx[n]$	$j\frac{dX(\Omega)}{d\Omega}$
Time reversal	$x[-n]$	$X(-\Omega)$
Time shifting	$x[n-k]$	$X(\Omega)e^{-jk\Omega}$ k integer
Frequency shifting	$x[n]e^{j\Omega_c n}$	$X(\Omega-\Omega_c)$
Time convolution	$x_1[n]*x_2[n]$	$X_1(\Omega)X_2(\Omega)$
Frequency convolution	$x_1[n]x_2[n]$	$\frac{1}{2\pi}\int_{2\pi}X_1[u]X_2[\Omega-u]\,du$
Parseval's theorem	$E_x=\sum_{-\infty}^{\infty}\|x[n]\|^2$	$E_x=\frac{1}{2\pi}\int_{2\pi}\|X(\Omega)\|^2\,d\Omega$