NATIONAL EXAMINATIONS DECEMBER 2010

04-BS-5 ADVANCED MATHEMATICS

3 Hours duration

NOTES:

1. If doubt exists as to the interpretation of any question, the candidate is urged to submit with the answer paper, a clear statement of any assumption made.

2. Candidates may use one of the approved Casio or Sharp calculators. This is a Closed Book Exam. However, candidates are permitted to bring **ONE** aid sheet (8.5"x11") written on both sides.

3. Any five (5) questions constitute a complete paper. Only the first five answers as they appear in your answer book will be marked.

4. All questions are of equal value.

Marking Scheme

1. 20 marks
2. (a) 14 marks ; (b) 6 marks
3. (a) 10 marks ; (b) 10 marks
4. (a) 12 marks ; (b) 8 marks
5. 20 marks
6. (a) 8 marks ; (b) 12 marks
7. (a) 10 marks ; (b) 10 marks

1.Consider the following differential equation:

$$(x^2-4)\frac{d^2y}{dx^2}+3x\frac{dy}{dx}+y=0$$

Find two linearly independent power series solutions about the ordinary point x=0.

2. (a) Find the Fourier series expansion of the periodic function F(x) of period p=2π.

$$F(x)=x^2 \quad ; \qquad 0 < x < 2\pi$$

(b) Use the result obtained in (a) to prove that

$$\frac{\pi^2}{6}=\sum_{n=1}^{\infty}\frac{1}{n^2}$$

3.(A) Prove that the coefficients α and β of the least-squares parabola $Y=\alpha X+\beta X^2$ that fits the set of n points (X_i, Y_i) can be obtained as follows

$$\alpha=\frac{\left\{\sum_{i=1}^{i=n}X_iY_i\right\}\left\{\sum_{i=1}^{i=n}X_i^4\right\}-\left\{\sum_{i=1}^{i=n}X_i^2Y_i\right\}\left\{\sum_{i=1}^{i=n}X_i^3\right\}}{\left\{\sum_{i=1}^{i=n}X_i^2\right\}\left\{\sum_{i=1}^{i=n}X_i^4\right\}-\left\{\sum_{i=1}^{i=n}X_i^3\right\}^2};$$

$$\beta=\frac{\left\{\sum_{i=1}^{i=n}X_i^2\right\}\left\{\sum_{i=1}^{i=n}X_i^2Y_i\right\}-\left\{\sum_{i=1}^{i=n}X_i^3\right\}\left\{\sum_{i=1}^{i=n}X_iY_i\right\}}{\left\{\sum_{i=1}^{i=n}X_i^2\right\}\left\{\sum_{i=1}^{i=n}X_i^4\right\}-\left\{\sum_{i=1}^{i=n}X_i^3\right\}^2}$$

3.(B) It has been suggested that the following set of n=7 points (X_i, Y_i) are related by an equation of the form $Y=\alpha+\beta X$. Use the method of least squares to find an estimate of the coefficients α and β.

X	0	1	2	3	4	5	6
Y	66	52	49	35	23	18	4

4.(A) Given the following data find Newton's interpolating polynomial of highest possible degree.

x	−4	−3	−2	−1	0	1	2	3	4
f(x)	570	111	−34	−45	−30	−25	6	171	650

4.(B) The following table displays the values of a certain function f(x) for a set of values of the independent variable x. Use the formulas given below the table to obtain an approximate value of the derivative of this function for x=–2,–1,0,1,2. Let $x_0 = -2$ and h = 1. Note that $f^{(1)}(x)$ denotes the first derivative of f(x).

x	–5	–4	–3	–2	–1	0	1	2	3	4	5
f(x)	–1559	–396	–793	–1100	–747	116	979	1332	1025	628	1791

$$f^{(1)}(x_0) \approx \frac{1}{12h}[-25f(x_0)+48f(x_0+h)-36f(x_0+2h)+16f(x_0+3h)-3f(x_0+4h)]$$

$$f^{(1)}(x_0+h) \approx \frac{1}{12h}[-3f(x_0)-10f(x_0+h)+18f(x_0+2h)-6f(x_0+3h)+f(x_0+4h)]$$

$$f^{(1)}(x_0+2h) \approx \frac{1}{12h}[f(x_0)-8f(x_0+h)+8f(x_0+3h)-f(x_0+4h)]$$

$$f^{(1)}(x_0+3h) \approx \frac{1}{12h}[-f(x_0)+6f(x_0+h)-18f(x_0+2h)+10f(x_0+3h)+3f(x_0+4h)]$$

$$f^{(1)}(x_0+4h) \approx \frac{1}{12h}[3f(x_0)-16f(x_0+h)+36f(x_0+2h)-48f(x_0+3h)+25f(x_0+4h)]$$

5. Use the Romberg algorithm with n = 2 to find the area bounded by $f(x)=(8+19x^2)^{2/3}$, x=0 , x=1 and y=0.
Hint: The Romberg algorithm produces a triangular array of numbers, all of which are numerical estimates of the definite integral $\int_a^b f(x)dx$. The array is denoted by the following notation:

R(0,0)		
R(1,0)	R(1,1)	
R(2,0)	R(2,1)	R(2,2)

where

$$R(0,0)=\frac{1}{2}(b-a)[f(a)+f(b)]$$

$$R(n,0)=\frac{1}{2}\ R(n-1,0)+h\sum_{k=1}^{2^{n-1}} f[a+(2k-1)h]$$

where $h=\dfrac{b-a}{2^n}$

$$R(n,m)=R(n,m-1)+\frac{1}{4^m-1}\ [R(n,m-1)-R(n-1,m-1)]$$

6.(A) One root of the equation $2e^{-x} - \sin x = 0$ lies between 0.8 and 1.0. Use the method of bisection five times to find a better approximation of this root. (Note: Carry five significant digits in your calculations).

6.(B) (i) One root of the equation $x^3 - 20x^2 + 110x - 123 = 0$ is close to $x_0 = 1.0$. Use the following iterative formula three times to find a better approximation of this root (Note: Carry seven significant digits in your calculations).

$$x_{i+1} = x_i - \frac{f(x_i)}{f^{(1)}(x_i) - \dfrac{f(x_i)f^{(2)}(x_i)}{2f^{(1)}(x_i)}}$$

[Hint: Let $f(x) = x^3 - 20x^2 + 110x - 123$. Note that $f^{(1)}(x)$ represents the first derivative of f(x). Similarly $f^{(2)}(x)$ represents the second derivative of f(x)].

(ii) Let r_1 be the root you obtained in (i). Find the other two roots by solving the quadratic equation $g(x) = 0$ where $g(x) = f(x)/(x - r_1)$.

7. The symmetric positive definite matrix $A = \begin{bmatrix} 9 & -6 & 9 \\ -6 & 20 & 14 \\ 9 & 14 & 38 \end{bmatrix}$ can be written as the product of a lower triangular matrix $L = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix}$ and its transpose L^T, that is $A = LL^T$.

(a) Find L and L^T.

(b) Use L and L^T to solve the following system of three linear equations:

$$\begin{aligned} 9x_1 - 6x_2 + 9x_3 &= 21 \\ -6x_1 + 20x_2 + 14x_3 &= -34 \\ 9x_1 + 14x_2 + 38x_3 &= 0 \end{aligned}$$