National Exams

December 2011

07-Elec-B1- Digital Signal Processing

3 Hours Duration

NOTES:

1. If doubt exists as to the interpretation of any question, the candidate is urged to submit with the answer paper, a clear statement of any assumptions made.
2. This is a Closed Book Exam, a Casio or Sharp approved calculator is permitted.
3. FOUR(4) questions constitute a complete paper. The first four questions as they appear in the answer book will be marked.
4. All questions are of equal value.
5. Clarity and organization of the answer are important.
6. LIST OF IMPORTANT EQUATIONS CAN BE FOUND ON LAST PAGE

1. **(25 marks total)** A digital filter is given by the system function

$$H(z) = \frac{z^{-1} - \alpha}{1 - \alpha z^{-1}},$$

where α is a real number.

(a) Prove that $H(z)$ is an all-pass system. (A system is an all-pass system if it has frequency response with unit magnitude, i.e., $|H(w)| = 1$).

(b) Determine the difference equation and impulse response of the filter.

(c) Determine the value of α, so that the filter is stable. Assume the filter is causal.

(d) Sketch the direct-form I and direct-form II realization of the filter.

2. **(25 marks total)** The following signal flow diagram represents a recursive system.

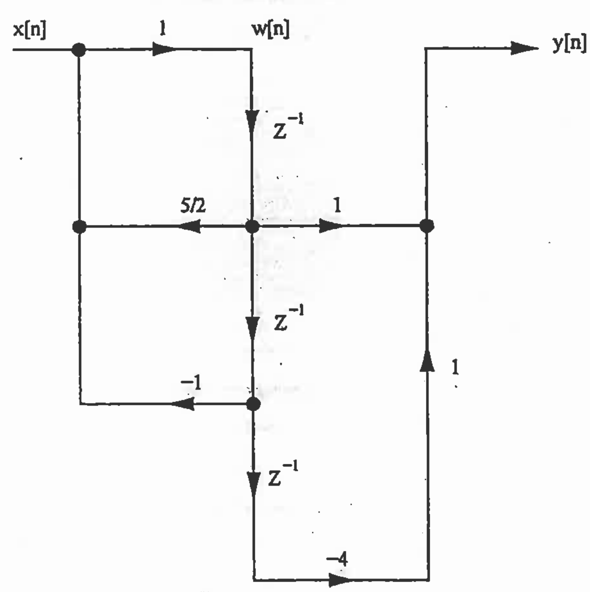


Figure 1:

(a) Determine the difference equation relating $y(n)$ to $x(n)$.

(b) Determine the system transfer function $H(z)$.

(c) Is the system stable? Justify your answer.

(d) Let $x(n) = (\frac{1}{3})^n u(n)$, where $u(n)$ is the unit step function. If all the initial conditions are zero, determine $y(n)$ for $n = 0, 1, 2, 3$.

3. **(25 marks total)** Consider the system described by the difference equation:

$$y[n] = x[n] + x[n-2] \tag{1}$$

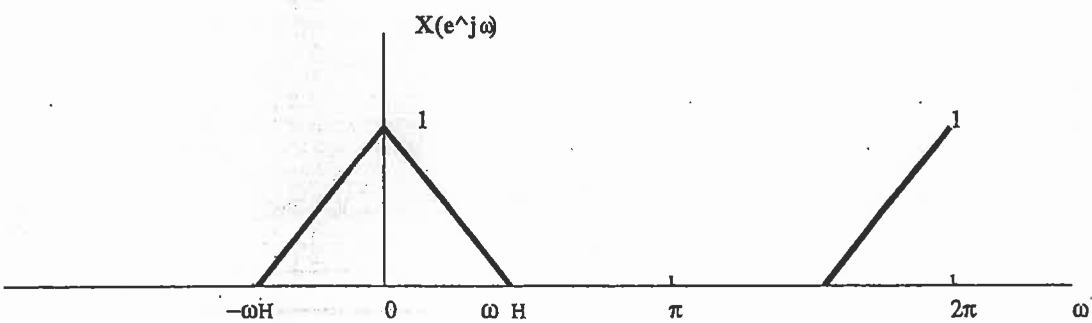
(a) Determine the impulse (unit sample) response sequence $h[n]$.

(b) Determine the frequency response function $H(e^{j\omega})$.

(c) Determine and sketch the magnitude response, $|H(e^{j\omega})|$, and the phase response, $\theta(\omega)$, functions of the system. Identify the digital frequencies (if any) that are completely blocked by the system.

(d) Determine the response of the system to the input $x[n] = \cos(\frac{\pi}{2}n + \frac{\pi}{4}) + \sin(\frac{\pi}{3}n)$.

4. **(25 marks total)** Consider the sequence $x[n]$ whose Fourier Transform $X(e^{j\omega})$ is shown in the below figure with the bandwidth of $(-\omega_H, \omega_H)$.



Define

$$x_s[n] = \begin{cases} x[n] & n = Mk \quad k = 0, \pm 1, \pm 2, ... \\ 0 & otherwise \end{cases} \tag{2}$$

and

$$x_d[n] = x_s[Mn] = x[Mn]$$

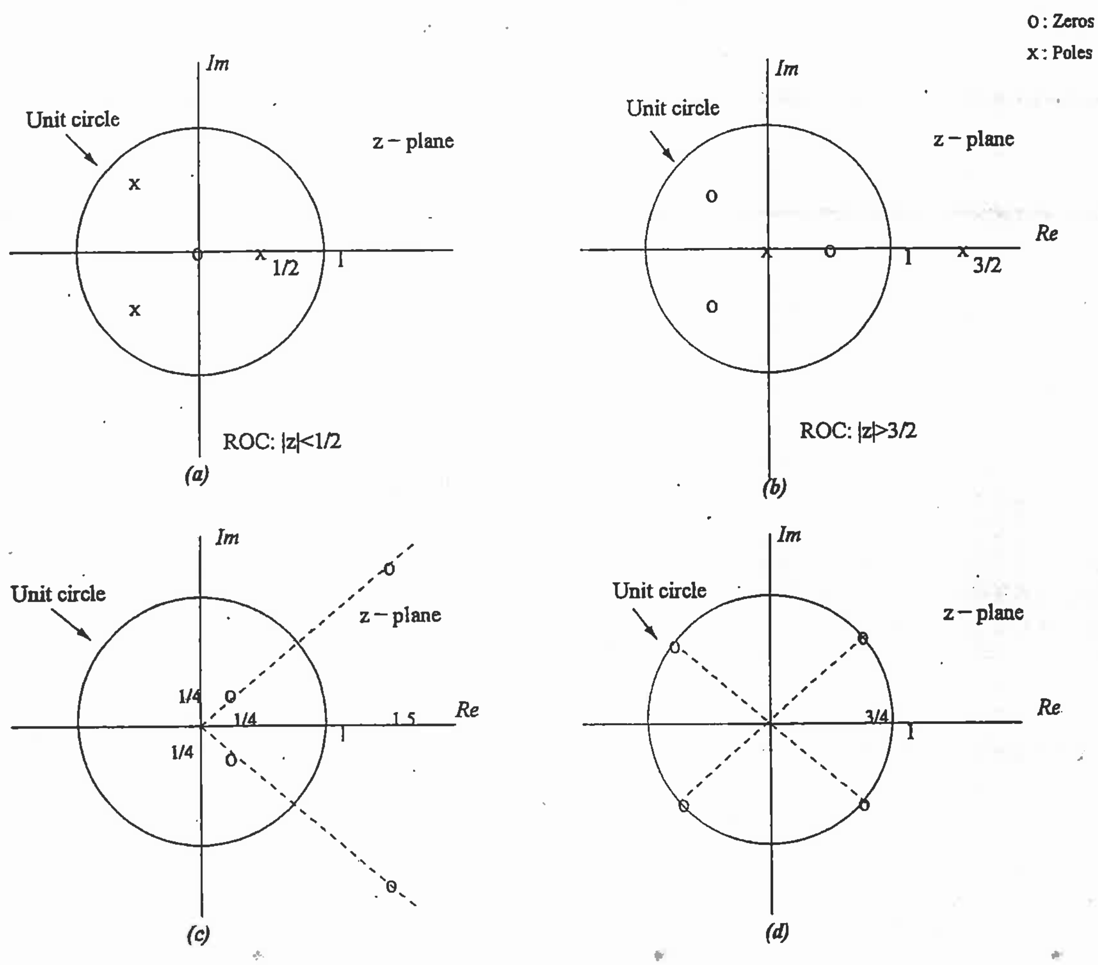
(a) (15pts) Sketch $X_s(e^{j\omega})$ for each of the following cases:

i. $M = 3$, $\omega_H = \pi/2$

ii. $M = 3$, $\omega_H = \pi/4$

(b) (10pts) What is the maximum value of ω_H that will avoid aliasing when $M = 3$?

5. (**25 marks total**) Each of the pole-zero plots in the following Figure, together with the specification of the region of convergence, describes a linear time-invariant system with system function $H(z)$. In each case, determine whether any of the following statements are true. Justify your answer with a brief statement or a counterexample.



(a) The system is zero-phase or a generalized linear-phase stystem.

(b) The system has a stable inverse $H_i(z)$.

Trigonometric Identities:

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$$
$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$

$$\sin\alpha\,\sin\beta = \tfrac{1}{2}\cos(\alpha-\beta) - \tfrac{1}{2}\cos(\alpha+\beta)$$
$$\cos\alpha\,\cos\beta = \tfrac{1}{2}\cos(\alpha+\beta) + \tfrac{1}{2}\cos(\alpha-\beta)$$
$$\sin\alpha\,\cos\beta = \tfrac{1}{2}\sin(\alpha+\beta) + \tfrac{1}{2}\sin(\alpha-\beta)$$

Euler's Identity:

$$e^{j\theta} = \cos\theta + j\sin\theta$$

Discrete Fourier Transform:

The N-point DFT of a N-sample sequence $s[n]$ is defined as:

$$S[k] = \sum_{n=0}^{N-1} s[n]\, W_N^{kn}, \qquad k = 0, 1, \ldots, N-1.$$

The sequence $s[n]$ can be recovered from its DFT coefficients using the N-point IDFT:

$$s[n] = \frac{1}{N}\sum_{k=0}^{N-1} S[k]\, W_N^{-kn}, \qquad n = 0, 1, \ldots, N-1.$$

where $W_N = e^{-j2\pi/N}$. The DFT/IDFT relations can also be expressed in matrix form

$$\begin{bmatrix} S[0] \\ \vdots \\ S[N-1] \end{bmatrix} = \mathbf{W}_N \begin{bmatrix} s[0] \\ \vdots \\ s[N-1] \end{bmatrix} \qquad \begin{bmatrix} s[0] \\ \vdots \\ s[N-1] \end{bmatrix} = \mathbf{W}_N^{-1} \begin{bmatrix} S[0] \\ \vdots \\ S[N-1] \end{bmatrix}$$

where the transformation matrices for $N = 2, 3, 4$ are

$$\mathbf{W}_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \qquad \mathbf{W}_2^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$\mathbf{W}_3 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -\frac{1}{2} - \frac{\sqrt{3}}{2}j & -\frac{1}{2} + \frac{\sqrt{3}}{2}j \\ 1 & -\frac{1}{2} + \frac{\sqrt{3}}{2}j & -\frac{1}{2} - \frac{\sqrt{3}}{2}j \end{bmatrix} \qquad \mathbf{W}_3^{-1} = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & -\frac{1}{2} + \frac{\sqrt{3}}{2}j & -\frac{1}{2} - \frac{\sqrt{3}}{2}j \\ 1 & -\frac{1}{2} - \frac{\sqrt{3}}{2}j & -\frac{1}{2} + \frac{\sqrt{3}}{2}j \end{bmatrix}$$

$$\mathbf{W}_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \qquad \mathbf{W}_4^{-1} = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & +j \end{bmatrix}$$

TABLE 9.1 A Short Table of Discrete-Time Fourier Transforms

No.	$x[n]$	$X(\Omega)$	
1	$\delta[n-k]$	$e^{-jk\Omega}$	Integer k
2	$\gamma^n u[n]$	$\dfrac{e^{j\Omega}}{e^{j\Omega}-\gamma}$	$\lvert\gamma\rvert < 1$
3	$-\gamma^n u[-(n+1)]$	$\dfrac{e^{j\Omega}}{e^{j\Omega}-\gamma}$	$\lvert\gamma\rvert > 1$
4	$\gamma^{\lvert n\rvert}$	$\dfrac{1-\gamma^2}{1-2\gamma\cos\Omega+\gamma^2}$	$\lvert\gamma\rvert < 1$
5	$n\gamma^n u[n]$	$\dfrac{\gamma e^{j\Omega}}{(e^{j\Omega}-\gamma)^2}$	$\lvert\gamma\rvert < 1$
6	$\gamma^n \cos(\Omega_0 n+\theta)u[n]$	$\dfrac{e^{j\Omega}[e^{j\Omega}\cos\theta-\gamma\cos(\Omega_0-\theta)]}{e^{j2\Omega}-(2\gamma\cos\Omega_0)e^{j\Omega}+\gamma^2}$	$\lvert\gamma\rvert < 1$
7	$u[n]-u[n-M]$	$\dfrac{\sin(M\Omega/2)}{\sin(\Omega/2)}e^{-j\Omega(M-1)/2}$	
8	$\dfrac{\Omega_c}{\pi}\operatorname{sinc}(\Omega_c n)$	$\displaystyle\sum_{k=-\infty}^{\infty}\operatorname{rect}\left(\frac{\Omega-2\pi k}{2\Omega_c}\right)$	$\Omega_c \le \pi$
9	$\dfrac{\Omega_c}{2\pi}\operatorname{sinc}^2\left(\dfrac{\Omega_c n}{2}\right)$	$\displaystyle\sum_{k=-\infty}^{\infty}\Delta\left(\frac{\Omega-2\pi k}{2\Omega_c}\right)$	$\Omega_c \le \pi$
10	$u[n]$	$\dfrac{e^{j\Omega}}{e^{j\Omega}-1}+\pi\displaystyle\sum_{k=-\infty}^{\infty}\delta(\Omega-2\pi k)$	
11	1 for all n	$2\pi\displaystyle\sum_{k=-\infty}^{\infty}\delta(\Omega-2\pi k)$	
12	$e^{j\Omega_0 n}$	$2\pi\displaystyle\sum_{k=-\infty}^{\infty}\delta(\Omega-\Omega_0-2\pi k)$	
13	$\cos\Omega_0 n$	$\pi\displaystyle\sum_{k=-\infty}^{\infty}\delta(\Omega-\Omega_0-2\pi k)+\delta(\Omega+\Omega_0-2\pi k)$	
14	$\sin\Omega_0 n$	$j\pi\displaystyle\sum_{k=-\infty}^{\infty}\delta(\Omega+\Omega_0-2\pi k)-\delta(\Omega-\Omega_0-2\pi k)$	
15	$(\cos\Omega_0 n)\,u[n]$	$\dfrac{e^{j2\Omega}-e^{j\Omega}\cos\Omega_0}{e^{j2\Omega}-2e^{j\Omega}\cos\Omega_0+1}+\dfrac{\pi}{2}\displaystyle\sum_{k=-\infty}^{\infty}\delta(\Omega-2\pi k-\Omega_0)+\delta(\Omega-2\pi k+\Omega_0)$	

TABLE 9.2 Properties of the DTFT

Operation	$x[n]$	$X(\Omega)$
Linearity	$a_1x_1[n]+a_2x_2[n]$	$a_1X_1(\Omega)+a_2X_2(\Omega)$
Conjugation	$x^*[n]$	$X^*(-\Omega)$
Scalar multiplication	$ax[n]$	$aX(\Omega)$
Multiplication by n	$nx[n]$	$j\dfrac{dX(\Omega)}{d\Omega}$
Time reversal	$x[-n]$	$X(-\Omega)$
Time shifting	$x[n-k]$	$X(\Omega)e^{-jk\Omega}$ k integer
Frequency shifting	$x[n]\,e^{j\Omega_c n}$	$X(\Omega-\Omega_c)$
Time convolution	$x_1[n]*x_2[n]$	$X_1(\Omega)X_2(\Omega)$
Frequency convolution	$x_1[n]x_2[n]$	$\dfrac{1}{2\pi}\displaystyle\int_{2\pi}X_1[u]X_2[\Omega-u]\,du$
Parseval's theorem	$E_x=\displaystyle\sum_{-\infty}^{\infty}\lvert x[n]\rvert^2$	$E_x=\dfrac{1}{2\pi}\displaystyle\int_{2\pi}\lvert X(\Omega)\rvert^2\,d\Omega$