National Exams December 2012
04-BS-1, Mathematics
3 hours Duration

Notes:

1. If doubt exists as to the interpretation of any question, the candidate is urged to submit with the answer paper, a clear statement of any assumptions made.
2. NO CALCULATOR is permitted. This is a CLOSED BOOK exam. However, candidates are permitted to bring ONE AID SHEET written on both sides.
3. Any five questions constitute a complete paper. Only the first five questions as they appear in your answer book will be marked.
4. All questions are of equal value.

Marking Scheme:

1. (a) 10 marks, (b) 10 marks
2. (a) 6 marks, (b) 6 marks, (c) 8 marks
3. (a) 7 marks, (b) 7 marks, (c) 6 marks
4. (a) 12 marks, (b) 8 marks
5. 20 marks
6. 20 marks
7. 20 marks
8. (a) 7 marks, (b) 6 marks, (c) 7 marks

1. (a) Solve the initial value problem

$$2y'' - 7y' + 3y = 10e^{3t}, \qquad y(0) = 0, \quad y'(0) = 4.$$

Note that $'$ denotes differentiation with respect to t.

(b) Find the general solution, $y(x)$, of the differential equation

$$2x^2y'' + xy' - y = 3x^2.$$

Note that $'$ now denotes differentiation with respect to x.

2. Let $\mathcal{P}$ be the plane passing through the three points (0,1,4), (1,1,3) and (0,2,2).

(a) Find an equation of the form $ax + by + cz = d$ for plane $\mathcal{P}$. (This is variously called the normal, general, or implicit equation for the plane.)

(b) Find a parametric representation for the plane $\mathcal{P}$. (This is often called the vector, or parametric, equation for the plane.)

(c) Find the line of intersection between the plane $\mathcal{P}$ and the plane

$$y + z = 1$$

3. Let $\mathcal{F}$ and $\mathcal{G}$ be the surfaces defined by the the equations

$$3x^2 + 2y^2 - 2z = 1$$

and

$$x^2 + y^2 + z^2 - 4y - 2z + 2 = 0$$

respectively.

(a) Find equations for the line perpendicular to $\mathcal{F}$ at the point $(1, 1, 2)$.

(b) Find an equation for the plane tangent to $\mathcal{G}$ at the point $(1, 1, 2)$.

(c) Find an equation for the the line tangent to the intersection of the surfaces at the point $(1, 1, 2)$.

4. Let $\mathcal{S}_1$ be the plane $2x + z + 4 = 0$ and $\mathcal{S}_2$ be the paraboloid $z = 4 - x^2 - 2y^2$.

(a) Set up the integral for the volume of the solid region above the plane $\mathcal{S}_1$ and below the paraboloid $\mathcal{S}_2$.

(b) Evaluate the integral from part (b). Hint, use the change of variables $x = 1 + r\cos\theta$, $y = (1/\sqrt{2})r\sin\theta$.

5. Find the maximum and minimum values of $f(x, y, z) = x + 2y - z$ over the ellipsoid $x^2 + y^2 + 3z^2 = 1$.

6. Find the work done by the field $\mathbf{F}(x, y, z) = x\mathbf{i} + y^2\mathbf{j} - 3z\mathbf{k}$ in moving a particle from the point $(0, 0, 2)$ to the point $(3\pi, -2, 0)$ along the path $x = 6t$, $y = -2\sin t$, $z = 2\cos t$.

7. Evaluate the surface integral $\iint_S \mathbf{F} \cdot dS$, where

$$\mathbf{F}(x, y, z) = 4x\mathbf{i} + 2x^2\mathbf{j} - 3\mathbf{k}$$

and S is the surface of the region bounded by the cone $z = 4 - \sqrt{x^2 - y^2}$ and the plane $z = 0$.

8. Let $x = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$, and let $A = \begin{pmatrix} 3 & -1 & 1 \\ -1 & 3 & -1 \\ 2 & -2 & 4 \end{pmatrix}$

   (a) Show that 2 is an eigenvalue of A and find an associated eigenvector.

   (b) Show that x is an eigenvector of A and find the associated eigenvalue.

   (c) Find the general solution to the differential equation $\dfrac{d\mathbf{y}}{dt} = A\mathbf{y}$.