National Exams May 2012

04-CHEM-B1, Transport Phenomena

3 hours duration

NOTES

1. If doubt exists as to the interpretation of any question, the candidate is urged to submit with the answer paper, a clear statement of any assumptions made.
2. The examination is an OPEN BOOK EXAM.
3. Candidates may use any **non-communicating** calculator.
4. **Not all** problems are of equal weight.
5. **Answer all four questions.**
6. State all assumptions clearly.
7. The various conservation equations (momentum, energy, and continuity) are given in Tables 1-4 appended to this paper taken from Brodkey, R.S. and Hershey H.C. (1988) *Transport Phenomena – A Unified Approach.*

Q1. [20 marks] Consider the system shown in Fig. 1, in which a cylindrical rod is being moved axially in the z-direction with velocity U. The rod and the cylinder are co-axial. The inner radius of the cylinder is R and the diameter of the rod κR such that $\kappa < 1$. Problems of this kind arise in the analysis of wire-coating dies such as those used in processing of thermoplastic materials.

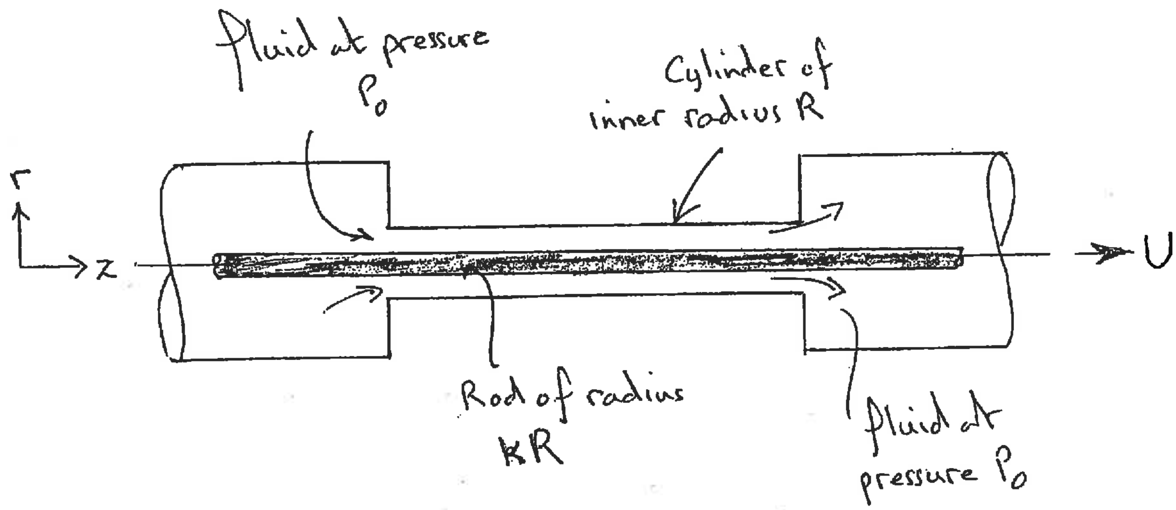

Fig. 1: Annular flow with inner cylinder moving axially

Starting with the appropriate form of the Navier-Stokes equations [*see Table 1 on p 5*], show that the steady-state velocity profile is given by:

$$u_z = U\frac{\ln(r/R)}{\ln(\kappa)}$$

Furthermore, show that the volumetric flow rate, $\dot{V}$, is given by:

$$\dot{V} = \frac{\pi R^2 U}{2}\left[\frac{1}{\ln(\kappa)}(\kappa^2 - 1) - 2\kappa^2\right]$$

Q2. [30 marks] A plane slab, of thickness $2L$ and thermal conductivity k, uniformly generates heat throughout at a rate $\dot{T}_G$. The temperatures at the left and right faces are T_1 (at $x = -L$) and T_2 (at $x = +L$). Starting with the appropriate form of the energy equation [*see Table 3 on p 7*], show that the steady-state temperature profile throughout the slab is:

$$T(x) = T_1 + \frac{\dot{T}_G}{2k}(L^2 - x^2) + \frac{(T_2 - T_1)}{2}\left(1 + \frac{x}{L}\right)$$

If both faces are in contact with air at T_∞ and the convective heat transfer coefficient at the left face is h_1, and that at the right face is h_2, develop expressions for the surface temperatures T_1 and T_2.

Q3. [20 marks] An important unit operation is the absorption of one of the constituents of a gas mixture preferentially in a contacting liquid. In some cases, depending upon the nature of the molecules, the absorption may or may not involve simultaneous chemical reaction. Consider a liquid surface being exposed to a gas mixture that contains a component A, which preferentially dissolves in the liquid B as shown in Fig. 2. At the liquid surface the concentration of A is C_{A_0}. Throughout the liquid A disappears according to a first-order chemical reaction with a specific reaction rate constant k_1. At some depth δ within the liquid the concentration of A will have reached zero.

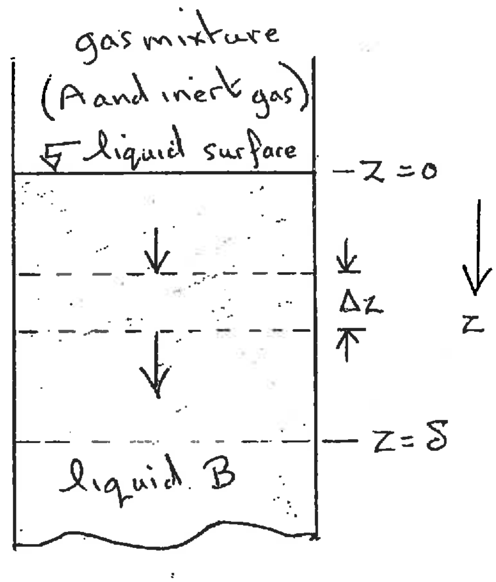

Fig. 2: Absorption of gas A in liquid b with chemical reaction

Starting with the appropriate form of the species continuity equation [*see Table 4 on p 7*], show that the concentration profile of A within the liquid B, is given by:

$$C_A = C_{A_0}\left\{\cosh(mz) - \frac{\sinh(mz)}{\tanh(m\delta)}\right\}$$

in which $m = \sqrt{k_1/D_{AB}}$.

Note: The general solution to differential equations of the form $ay - b\frac{d^2y}{dx^2} = 0$ is given by

$$y = C_1\cosh\left[\sqrt{a/b}\cdot x\right] + C_2\sinh\left[\sqrt{a/b}\cdot x\right]$$

Q4. [30 marks] Figure 3 shows a liquid falling as a thin film under laminar flow down a vertical flat surface whilst being exposed to a gas A, which dissolves in the liquid. The liquid contains a uniform concentration of $A\left(C_{A_0}\right)$ at the top (z = 0), and at the liquid surface (x = 0) the concentration of gas is $\left(C_{A_i}\right)$.

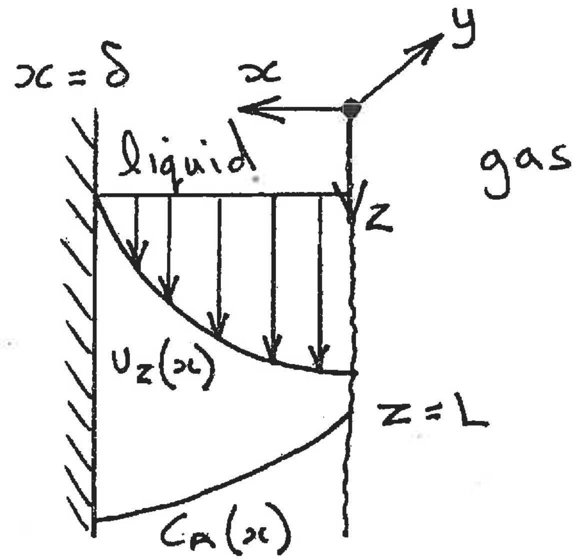

Fig. 3: Falling liquid film

Since this is a problem of fluid flow and mass transfer, first select the appropriate form of the Navier-Stokes equation [*see Table 1 on p 5*] and develop an expression for the applicable velocity distribution. Then select the appropriate form of the continuity equation for species A [*see Table 4 on p 7*], make the necessary simplifying assumptions and utilizing the resultant velocity distribution show that the governing differential equation for the problem is:

$$\left\{\rho g_z \delta^2 \frac{\left[1-(x/\delta)^2\right]}{2\mu}\right\}\frac{\partial C_A}{\partial z} = D\frac{\partial^2 C_A}{\partial x^2}$$

Define the boundary conditions but do not attempt to solve the equation.

Table 1: The Navier-Stokes equations for fluids of constant ρ and μ[1]

Navier–Stokes equation in vector form (rectangular coordinates only)

$$\partial U/\partial t + (U\cdot\nabla)U = -(1/\rho)\nabla p + g + \nu(\nabla^2 U) \qquad (5.15)$$

Rectangular coordinates

x component:
$$\frac{\partial U_x}{\partial t} + U_x\frac{\partial U_x}{\partial x} + U_y\frac{\partial U_x}{\partial y} + U_z\frac{\partial U_x}{\partial z} = -\frac{1}{\rho}\left(\frac{\partial p}{\partial x}\right) + g_x + \nu\left(\frac{\partial^2 U_x}{\partial x^2} + \frac{\partial^2 U_x}{\partial y^2} + \frac{\partial^2 U_x}{\partial z^2}\right) \qquad (A)$$

y component:
$$\frac{\partial U_y}{\partial t} + U_x\frac{\partial U_y}{\partial x} + U_y\frac{\partial U_y}{\partial y} + U_z\frac{\partial U_y}{\partial z} = -\frac{1}{\rho}\left(\frac{\partial p}{\partial y}\right) + g_y + \nu\left(\frac{\partial^2 U_y}{\partial x^2} + \frac{\partial^2 U_y}{\partial y^2} + \frac{\partial^2 U_y}{\partial z^2}\right) \qquad (B)$$

z component:
$$\frac{\partial U_z}{\partial t} + U_x\frac{\partial U_z}{\partial x} + U_y\frac{\partial U_z}{\partial y} + U_z\frac{\partial U_z}{\partial z} = -\frac{1}{\rho}\left(\frac{\partial p}{\partial z}\right) + g_z + \nu\left(\frac{\partial^2 U_z}{\partial x^2} + \frac{\partial^2 U_z}{\partial y^2} + \frac{\partial^2 U_z}{\partial z^2}\right) \qquad (C)$$

Cylindrical coordinates

r component:
$$\frac{\partial U_r}{\partial t} + U_r\frac{\partial U_r}{\partial r} + \frac{U_\theta}{r}\frac{\partial U_r}{\partial \theta} + U_z\frac{\partial U_r}{\partial z} - \frac{U_\theta^2}{r}$$
$$= -\frac{1}{\rho}\left(\frac{\partial p}{\partial r}\right) + g_r + \nu\frac{\partial^2 U_r}{\partial r^2} + \frac{\nu}{r}\frac{\partial U_r}{\partial r} - \nu\left(\frac{U_r}{r^2}\right) + \frac{\nu}{r^2}\frac{\partial^2 U_r}{\partial \theta^2} - \frac{2\nu}{r^2}\frac{\partial U_\theta}{\partial \theta} + \nu\frac{\partial^2 U_r}{\partial z^2} \qquad (D)$$

θ component:
$$\frac{\partial U_\theta}{\partial t} + U_r\frac{\partial U_\theta}{\partial r} + \frac{U_\theta}{r}\frac{\partial U_\theta}{\partial \theta} + U_z\frac{\partial U_\theta}{\partial z} + \frac{U_r U_\theta}{r}$$
$$= -\frac{1}{\rho r}\frac{\partial p}{\partial \theta} + g_\theta + \nu\frac{\partial^2 U_\theta}{\partial r^2} + \frac{\nu}{r}\frac{\partial U_\theta}{\partial r} - \nu\left(\frac{U_\theta}{r^2}\right) + \frac{\nu}{r^2}\frac{\partial^2 U_\theta}{\partial \theta^2} + \frac{2\nu}{r^2}\frac{\partial U_r}{\partial \theta} + \nu\frac{\partial^2 U_\theta}{\partial z^2} \qquad (E)$$

z component:
$$\frac{\partial U_z}{\partial t} + U_r\frac{\partial U_z}{\partial r} + \frac{U_\theta}{r}\frac{\partial U_z}{\partial \theta} + U_z\frac{\partial U_z}{\partial z}$$
$$= -\frac{1}{\rho}\frac{\partial p}{\partial z} + g_z + \nu\frac{\partial^2 U_z}{\partial r^2} + \frac{\nu}{r}\frac{\partial U_z}{\partial r} + \frac{\nu}{r^2}\frac{\partial^2 U_z}{\partial \theta^2} + \nu\frac{\partial^2 U_z}{\partial z^2} \qquad (F)$$

Spherical coordinates

r component:
$$\frac{\partial U_r}{\partial t} + U_r\frac{\partial U_r}{\partial r} + \frac{U_\theta}{r}\frac{\partial U_r}{\partial \theta} + \left(\frac{U_\phi}{r\sin\theta}\right)\frac{\partial U_r}{\partial \phi} - \frac{U_\theta^2}{r} - \frac{U_\phi^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + g_r + \frac{\nu}{r^2}\left(\frac{\partial}{\partial r}\left(r^2\frac{\partial U_r}{\partial r}\right)\right)$$
$$+\left(\frac{\nu}{r^2\sin\theta}\right)\left(\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial U_r}{\partial \theta}\right)\right) + \left(\frac{\nu}{r^2\sin^2\theta}\right)\left(\frac{\partial^2 U_r}{\partial \phi^2}\right) - \frac{2\nu U_r}{r^2}$$
$$-\frac{2\nu}{r^2}\frac{\partial U_\theta}{\partial \theta} - \frac{2\nu U_\theta}{r^2}\cot\theta - \left(\frac{2\nu}{r^2\sin\theta}\right)\frac{\partial U_\phi}{\partial \phi} \qquad (G)$$

θ component:
$$\frac{\partial U_\theta}{\partial t} + U_r\frac{\partial U_\theta}{\partial r} + \frac{U_\theta}{r}\frac{\partial U_\theta}{\partial \theta} + \left(\frac{U_\phi}{r\sin\theta}\right)\left(\frac{\partial U_\theta}{\partial \phi}\right) + \frac{U_r U_\theta}{r} - \frac{U_\phi^2}{r}\cot\theta$$
$$= -\frac{1}{\rho r}\frac{\partial p}{\partial \theta} + g_\theta + \frac{\nu}{r^2}\left(\frac{\partial}{\partial r}\left(r^2\frac{\partial U_\theta}{\partial r}\right)\right) + \left(\frac{\nu}{r^2\sin\theta}\right)\left(\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial U_\theta}{\partial \theta}\right)\right) + \left(\frac{\nu}{r^2\sin^2\theta}\right)\frac{\partial^2 U_\theta}{\partial^2 \phi}$$
$$+\frac{2\nu}{r^2}\frac{\partial U_r}{\partial \theta} - \left(\frac{\nu U_\theta}{r^2\sin^2\theta}\right) - \left(\frac{(2\nu)\cos\theta}{r^2\sin^2\theta}\right)\frac{\partial U_\phi}{\partial \phi} \qquad (H)$$

φ component:
$$\frac{\partial U_\phi}{\partial t} + U_r\frac{\partial U_\phi}{\partial r} + \frac{U_\theta}{r}\frac{\partial U_\phi}{\partial \theta} + \left(\frac{U_\phi}{r\sin\theta}\right)\frac{\partial U_\phi}{\partial \phi} + \frac{U_r U_\phi}{r} + \frac{U_\theta U_\phi}{r}\cot\theta$$
$$= -\left(\frac{1}{\rho r\sin\theta}\right)\frac{\partial p}{\partial \phi} + g_\phi + \frac{\nu}{r^2}\left(\frac{\partial}{\partial r}\left(r^2\frac{\partial U_\phi}{\partial r}\right)\right) + \left(\frac{\nu}{r^2\sin\theta}\right)\left(\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial U_\phi}{\partial \theta}\right)\right)$$
$$+\left(\frac{\nu}{r^2\sin^2\theta}\right)\frac{\partial^2 U_\phi}{\partial \phi^2} - \left(\frac{\nu U_\phi}{r^2\sin^2\theta}\right) + \left(\frac{2\nu}{r^2\sin\theta}\right)\frac{\partial U_r}{\partial \phi} + \left(\frac{2\nu\cos\theta}{r^2\sin^2\theta}\right)\frac{\partial U_\theta}{\partial \phi} \qquad (I)$$

[1] Brodkey, R.S. and Hershey H.C. (1988) *Transport Phenomena – A Unified Approach* Table 5.7 p147.

Table 2: Shear stress-velocity gradient relationships for constant viscosity[2]

Rectangular coordinates

$$\tau_{xx} = -2\mu(\partial U_x/\partial x) + (2\mu/3)(\nabla \cdot U) \quad \text{(A)}$$

$$\tau_{yy} = -2\mu(\partial U_y/\partial y) + (2\mu/3)(\nabla \cdot U) \quad \text{(B)}$$

$$\tau_{zz} = -2\mu(\partial U_z/\partial z) + (2\mu/3)(\nabla \cdot U) \quad \text{(C)}$$

$$\tau_{xy} = \tau_{yx} = -\mu[(\partial U_x/\partial y) + (\partial U_y/\partial x)] \quad \text{(D)}$$

$$\tau_{yz} = \tau_{zy} = -\mu[(\partial U_y/\partial z) + (\partial U_z/\partial y)] \quad \text{(E)}$$

$$\tau_{xz} = \tau_{zx} = -\mu[(\partial U_x/\partial z) + (\partial U_z/\partial x)] \quad \text{(F)}$$

Cylindrical coordinates

$$\tau_{rr} = -2\mu(\partial U_r/\partial r) + (2\mu/3)(\nabla \cdot U) \quad \text{(G)}$$

$$\tau_{\theta\theta} = -2\mu\left[\frac{1}{r}\left(\frac{\partial U_\theta}{\partial \theta}\right) + \frac{U_r}{r}\right] + (2\mu/3)(\nabla \cdot U) \quad \text{(H)}$$

$$\tau_{zz} = -2\mu(\partial U_z/\partial z) + (2\mu/3)(\nabla \cdot U) \quad \text{(I)}$$

$$\tau_{r\theta} = \tau_{\theta r} = -\mu\left[r\frac{\partial}{\partial r}(U_\theta/r) + \frac{1}{r}\left(\frac{\partial U_r}{\partial \theta}\right)\right] \quad \text{(J)}$$

$$\tau_{\theta z} = \tau_{z\theta} = -\mu\left[\left(\frac{\partial U_\theta}{\partial z}\right) + \frac{1}{r}\left(\frac{\partial U_z}{\partial \theta}\right)\right] \quad \text{(K)}$$

$$\tau_{rz} = \tau_{zr} = -\mu[(\partial U_r/\partial z) + (\partial U_z/\partial r)] \quad \text{(L)}$$

Spherical coordinates

$$\tau_{rr} = -2\mu(\partial U_r/\partial r) + (2\mu/3)(\nabla \cdot U) \quad \text{(M)}$$

$$\tau_{\theta\theta} = -2\mu\left[\frac{1}{r}\left(\frac{\partial U_\theta}{\partial \theta}\right) + \frac{U_r}{r}\right] + (2\mu/3)(\nabla \cdot U) \quad \text{(N)}$$

$$\tau_{\phi\phi} = -2\mu\left[\frac{1}{r\sin\theta}\frac{\partial U_\phi}{\partial \phi} + \frac{U_r}{r} + (U_\theta/r)(\cot\theta)\right] + (2\mu/3)(\nabla \cdot U) \quad \text{(O)}$$

$$\tau_{r\theta} = \tau_{\theta r} = -\mu\left[r\frac{\partial}{\partial r}(U_\theta/r) + \frac{1}{r}\left(\frac{\partial U_r}{\partial \theta}\right)\right] \quad \text{(P)}$$

$$\tau_{\theta\phi} = \tau_{\phi\theta} = -\mu\left[\frac{\sin\theta}{r}\left[\frac{\partial}{\partial \theta}\left(\frac{U_\phi}{\sin\theta}\right)\right] + \frac{1}{r\sin\theta}\frac{\partial U_\theta}{\partial \phi}\right] \quad \text{(Q)}$$

$$\tau_{r\phi} = \tau_{\phi r} = -\mu\left[\frac{1}{r\sin\theta}\frac{\partial U_r}{\partial \phi} + r\frac{\partial}{\partial r}(U_\phi/r)\right] \quad \text{(R)}$$

[2] B&H *ibid* Table 5.2 p137.

Table 3: The energy equation[3]

General equation

$$\frac{\partial(\rho c_p T)}{\partial t} + (U \cdot \nabla)(\rho c_p T) = \dot{T}_G + [\nabla \cdot \alpha \nabla(\rho c_p T)] - (\rho c_p T)(\nabla \cdot U) \qquad (5.13)$$

Incompressible media, rectangular coordinates

$$\frac{\partial T}{\partial t} + U_x \frac{\partial T}{\partial x} + U_y \frac{\partial T}{\partial y} + U_z \frac{\partial T}{\partial z} = \frac{\dot{T}_G}{\rho c_p} + \frac{\partial}{\partial x}\left(\alpha \frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(\alpha \frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(\alpha \frac{\partial T}{\partial z}\right) \qquad (A)$$

Incompressible media, cylindrical coordinates

$$\frac{\partial T}{\partial t} + U_r \frac{\partial T}{\partial r} + \frac{U_\theta}{r}\frac{\partial T}{\partial \theta} + U_z \frac{\partial T}{\partial z} = \frac{\dot{T}_G}{\rho c_p} + \frac{1}{r}\frac{\partial}{\partial r}\left(r\alpha \frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial \theta}\left(\alpha \frac{\partial T}{\partial \theta}\right) + \frac{\partial}{\partial z}\left(\alpha \frac{\partial T}{\partial z}\right) \qquad (B)$$

Incompressible media, spherical coordinates

$$\frac{\partial T}{\partial t} + U_r \frac{\partial T}{\partial r} + \frac{U_\theta}{r}\frac{\partial T}{\partial \theta} + \frac{U_\phi}{r \sin\theta}\frac{\partial T}{\partial \phi} = \frac{\dot{T}_G}{\rho c_p} + \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \alpha \frac{\partial T}{\partial r}\right)$$

$$+ \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\alpha \sin\theta \frac{\partial T}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial}{\partial \phi}\left(\alpha \frac{\partial T}{\partial \phi}\right) \qquad (C)$$

Table 4: The continuity equation for species A[4]

General equation

$$\frac{\partial C_A}{\partial t} + (U \cdot \nabla)C_A = \dot{C}_{A,G} + (\nabla \cdot D\nabla C_A) - (C_A)(\nabla \cdot U) \qquad (5.8)$$

Incompressible media, rectangular coordinates

$$\frac{\partial C_A}{\partial t} + U_x \frac{\partial C_A}{\partial x} + U_y \frac{\partial C_A}{\partial y} + U_z \frac{\partial C_A}{\partial z} = \dot{C}_{A,G} + \frac{\partial}{\partial x}\left(D \frac{\partial C_A}{\partial x}\right) + \frac{\partial}{\partial y}\left(D \frac{\partial C_A}{\partial y}\right) + \frac{\partial}{\partial z}\left(D \frac{\partial C_A}{\partial z}\right) \qquad (A)$$

Incompressible media, cylindrical coordinates

$$\frac{\partial C_A}{\partial t} + U_r \frac{\partial C_A}{\partial r} + \frac{U_\theta}{r}\frac{\partial C_A}{\partial \theta} + U_z \frac{\partial C_A}{\partial z} = \dot{C}_{A,G} + \frac{1}{r}\frac{\partial}{\partial r}\left(rD \frac{\partial C_A}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial \theta}\left(D \frac{\partial C_A}{\partial \theta}\right) + \frac{\partial}{\partial z}\left(D \frac{\partial C_A}{\partial z}\right) \qquad (B)$$

Incompressible media, spherical coordinates

$$\frac{\partial C_A}{\partial t} + U_r \frac{\partial C_A}{\partial r} + \frac{U_\theta}{r}\frac{\partial C_A}{\partial \theta} + \frac{U_\phi}{r \sin\theta}\frac{\partial C_A}{\partial \phi} = \dot{C}_{A,G} + \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 D \frac{\partial C_A}{\partial r}\right)$$

$$+ \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(D \sin\theta \frac{\partial C_A}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial}{\partial \phi}\left(D \frac{\partial C_A}{\partial \phi}\right) \qquad (C)$$

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[3] B&H *ibid* Table 5.6 p143.

[4] B&H *ibid* Table 5.4 p142.