

## National Exams May 2014

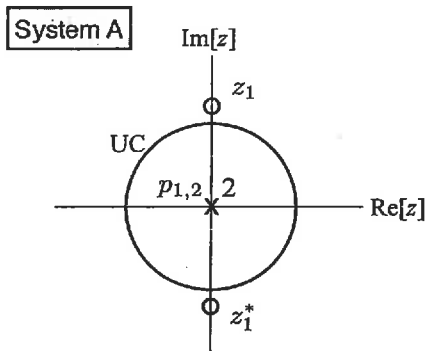
**07-Elec-B1, Digital Signal Processing**

3 hours duration

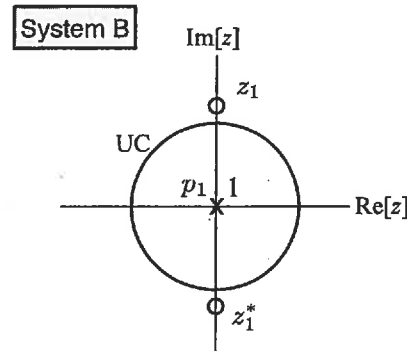
**NOTES:**

1. If doubt exists as to the interpretation of any question, the candidate is urged to submit with the answer paper, a clear statement of any assumptions made.
2. This is an **OPEN BOOK EXAM**. Any non-communicating calculator is permitted.
3. **Five (5)** questions constitute a complete paper. The first five questions as they appear in your answer book will be marked.
4. All questions are of equal value.
5. You can find a list of commonly used symbols, some trigonometric identities, Discrete-Fourier Transform definitions and formulations, Discrete-Time Fourier Transform tables and  $z$ -Transform tables at the end of this exam paper.

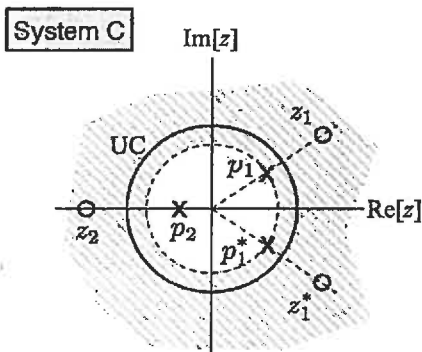
1. Figure (1) depicts the pole-zero diagrams corresponding to the transfer functions of four independent systems. *Note: In all diagrams UC represents the unit circle. The region-of-convergence of each system is indicated below the corresponding pole-zero diagram.*



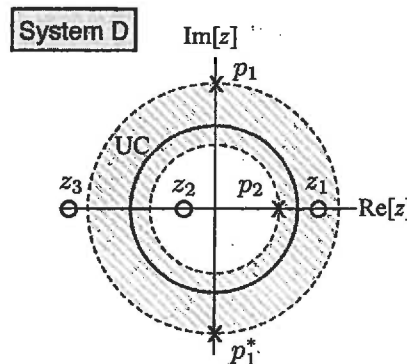
$$\text{ROC} = \{z : \text{entire } z\text{-plane except } z = 0\}$$



$$\text{ROC} = \{z : \text{entire } z\text{-plane except } z = 0 \text{ and } z = \infty\}$$



$$\text{ROC} = \{z : |z| > |p_1|\}$$



$$\text{ROC} = \{z : |p_2| < |z| < |p_1|\}$$

**Figure 1:** Pole-zero diagrams of four different systems.

- (a) For each system determine the following system characteristics:

- Type of system (FIR or IIR);
- Stability;
- Causality.

Explain and justify your answers. Answers without explanations will not receive any credit.

- (b) For each system determine the whether the corresponding unit impulse response sequence  $h[n]$  is a double-sided, left-sided, right-sided, mixed or finite length sequence.

2. Let  $x(t)$  be a real-valued analog signal applied to the input of the A/D which operates at the sampling rate  $F_s = 48$  kHz. The input signal  $x(t)$  is corrupted by two strong real-valued sinusoids with frequencies  $F_1$  and  $F_2$ , such that  $0 < F_1 < F_2 < 24$  kHz. Figure (2) displays the magnitude spectrum of the 64-point DFT obtained from the 48 kHz sampled sequence  $x[n]$ .

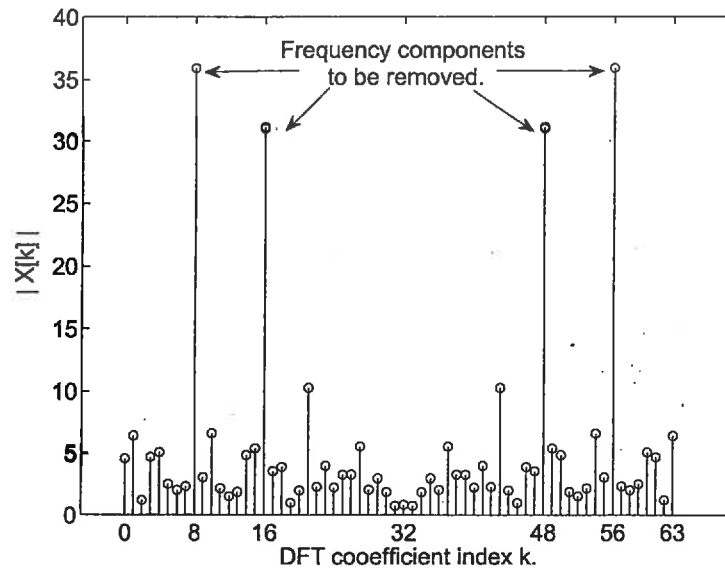


Figure 2: Magnitude spectrum of 64-point DFT of  $x[n]$ .

- From the magnitude spectrum shown in Figure (2) identify the frequencies  $F_1$  and  $F_2$  of the two real-valued sinusoids.
- Design a causal FIR notch filter with real-valued coefficients which would remove the unwanted sinusoids at frequencies  $F_1$  and  $F_2$ . Determine the system function,  $H_{notch}(z)$ , of the filter.
- Sketch the pole-zero diagram of  $H_{notch}(z)$ . Sketch a signal-flow/block diagram which implements the notch filter described by  $H_{notch}(z)$  using a minimum number of multiplications. Determine the number of additions/subtractions, multiplications and storage requirements for this implementation.
- If the input to  $H_{notch}(z)$  has been sampled at 10 kHz, which frequencies will be blocked by the filter?
- Convert the FIR filter  $H_{notch}(z)$  into an IIR filter for improved performance. The IIR filter should again remove the *d.c.* component and the sinusoids at frequencies  $F_1$  and  $F_2$ , while reducing the bandwidth of the notches in the magnitude response of the filter. Determine the transfer function of this IIR filter and provide a signal-flow/block diagram representation.

3. Let  $\mathbf{x} = \{x[0], \dots, x[N-1]\}$  be an  $N$ -element sequence. Let  $\mathbf{X} = \{X[0], \dots, X[N-1]\}$  be the corresponding  $N$ -point DFT of  $\mathbf{x}$  such that

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn}, \quad k = 0, 1, \dots, N-1; \quad (1)$$

where  $W_N = e^{-j2\pi/N}$ . An engineer wants to recover  $\mathbf{x}$  by applying the inverse transform IDFT to  $\mathbf{X}$ . However, as a result of carelessness the engineer applies DFT to  $\mathbf{X}$  again and generates the sequence  $\mathbf{y} = \{y[0], \dots, y[N-1]\}$ :

$$y[m] = \sum_{k=0}^{N-1} X[k] W_N^{mk}, \quad m = 0, 1, \dots, N-1; \quad (2)$$

Your task is to help the engineer by exploring if it is still possible to recover  $\mathbf{x}$  from  $\mathbf{y}$ .

- (a) If  $\mathbf{x} = \{0, 1, -1, 2\}$ , determine  $\mathbf{y}$ . Does it appear that  $\mathbf{x}$  can be recovered from  $\mathbf{y}$ ?  
 (b) Prove that  $\mathbf{x}$  can be recovered from  $\mathbf{y}$  for the general case when  $\mathbf{x}$  is an arbitrary  $N$ -element sequence. If  $X[k]$  and  $y[m]$  are given by Equations (1) and (2), respectively, determine  $\mathbf{y}$  in terms of  $\mathbf{x}$ . **Hint:** Consider using the relation:

$$\frac{1}{N} \sum_{k=0}^{N-1} W_N^{(n-l)k} = \begin{cases} 1, & n = \langle l \rangle_N; \\ 0, & \text{otherwise.} \end{cases} \quad (3)$$

to simplify the expressions, where the notation  $\langle \cdot \rangle$  denotes the *modulo* operation:

$$\langle l \rangle_N = l \text{ modulo } N \quad (4)$$

- (c) Using the relation you derived in part (b), show how the elements of  $\mathbf{y}$  are related to the elements of  $\mathbf{x} = \{x[0], x[1], \dots, x[15]\}$ .  
 (d) If the DFT operations used in transforming  $\mathbf{x}$  into  $\mathbf{X}$ , and  $\mathbf{X}$  into  $\mathbf{y}$  are implemented **directly** using Equations (1) and (2), respectively, determine the total number of real multiplications needed to generate  $\mathbf{y}$  from  $\mathbf{x}$ . Also determine the total number of real multiplications needed if the DFTs are implemented using a radix-2 FFT algorithm. Assume  $N = 2^q$  with  $q \in \mathbb{Z}^+$ .

4. Consider the discrete-time, linear, time-invariant system  $\mathcal{S}$  described by the difference equation:

$$y[n] = x[n] - x[n - 1]. \quad (5)$$

Let the input to the system be the 3-sample sequence

$$x[n] = \left\{ \underset{\substack{\uparrow \\ n=0}}{1}, 0, -1 \right\} \quad (6)$$

- (a) Determine the unit impulse response sequence  $h[n]$  that describes the system  $\mathcal{S}$ .
- (b) Determine the system output  $y[n]$  when  $x[n]$  is the input using the **iterative** solution of the difference equation given in Equation (5).
- (c) Determine  $y[n]$  using **linear convolution**.
- (d) Determine  $y[n]$  using **circular convolution** (implemented in time domain).
- (e) Determine  $y[n]$  using **DFT/IDFT** operations.

5. Let  $\mathcal{S}$  be a discrete-time, linear, time-invariant system which is known to be **causal** and **stable**. The system  $\mathcal{S}$  is described by the transfer function  $H(z)$  with the following characteristics:
- System Zeros:  $z_1 = 0$  and  $z_2 = -1$ ;
  - System Poles:  $p_1 = 0.5$  and  $p_2 = -0.5$ ;
  - $H(1) = \text{d.c. gain} = 8/3$ .
- (a) Determine the **transfer function**  $H(z)$  and its **region-of-convergence**.
- (b) Determine the **difference equation** that describes the system.
- (c) Determine the **unit impulse response sequence**  $h[n]$ .
- (d) Let  $x[n] = (\frac{1}{3})^n u[n]$  be the input to the system where  $u[n]$  is the unit step sequence. The initial conditions for the system are  $y[-1] = 0$  and  $y[-2] = 8$ . Determine the **total system response**  $y[n]$ . You may use any method of your choice.
- (e) Sketch a **canonic realization** of the system that uses minimum number of delay elements. Determine the number of additions, multiplications and delay elements used by your realization.

6. Let  $\mathcal{S}$  be a discrete-time, linear, time-invariant system which is known to be **causal** and **stable**. The system  $\mathcal{S}$  is described by the difference equation:

$$y[n] = \left(\frac{1}{2}\right)y[n-1] + x[n] + \left(\frac{1}{2}\right)x[n-1]. \quad (7)$$

- (a) Determine the impulse (unit sample) response sequence  $h[n]$ .
- (b) Determine the frequency response function  $H(e^{j\omega})$ .
- (c) Determine the response of the system to the input  $x[n] = \cos\left(\frac{\pi}{2}n + \frac{\pi}{4}\right)$ .

7. In this question we investigate how we can use the DFT algorithm to calculate the inverse-DFT (IDFT). The box labeled “N-point DFT” shown in Figure (3), operates on the input sequence  $\{x[n]\}_{n=0}^{N-1}$  and generates the N-point DFT sequence  $\{X[k]\}_{k=0}^{N-1}$ , where  $\mathbf{Re}[z]$  and  $\mathbf{Im}[z]$  refer to the *real* and *imaginary* parts of  $z$ , respectively.

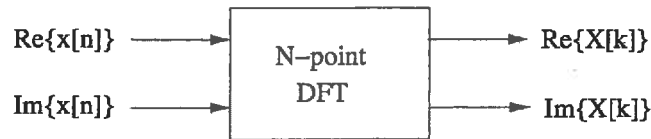


Figure 3: Representation of the N-point DFT block.

- (a) Determine the modifications that are **external** to the “box”, such that when the input to the modified system is  $\{X[k]\}_{k=0}^{N-1}$  then the output of the modified system will be  $\{x[n]\}_{n=0}^{N-1}$ .
- (b) A second approach to the IDFT computation using a DFT algorithm is illustrated in Figure (4).

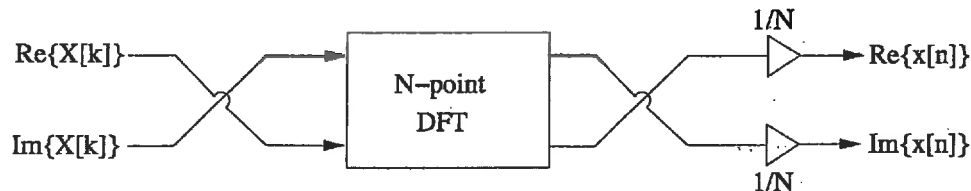


Figure 4: Computing the IDFT from DFT using Equation (9).

Define a length-N sequence  $\{q[n]\}_{n=0}^{N-1}$  as

$$\mathbf{Re}[q[n]] = \mathbf{Im}X[k] \Big|_{k=n}, \quad \mathbf{Im}[q[n]] = \mathbf{Re}X[k] \Big|_{k=n}. \quad (8)$$

with  $Q[k]$  denoting its N-point DFT. Demonstrate that this approach will indeed allow you to calculate the IDFT by showing that

$$\mathbf{Re}[x[n]] = \frac{1}{N} \mathbf{Im} Q[k] \Big|_{k=n}, \quad \mathbf{Im}[x[n]] = \frac{1}{N} \mathbf{Re} Q[k] \Big|_{k=n}. \quad (9)$$

- (c) Let  $X[k] = \{1, 1 + 2j, 1, 1 - 2j\}$  be the 4-point DFT of the time-domain sequence  $x[n]$ . Determine  $x[n]$  by evaluating the IDFT of  $X[k]$  using the approach delineated in part (b). (No credit will be given if  $x[n]$  is determined using an approach different than the one described in part (b).)



**List of Commonly Used Symbols:**

$a(t), x(t), y(t), \dots$	continuous-time signals.
$a[n], x[n], y[n], \dots$	discrete-time sequences.
$A(z)$	$z$ -Transform of the discrete-time sequence $a[n]$
$h[n]$	impulse response sequence of a discrete-time system.
$H(z)$	transfer function of a discrete-time linear, time-invariant system represented by the impulse response sequence $h[n]$ .
$\Omega$	angular frequency.
$F$	frequency such that $\Omega = 2\pi F$ .
$\omega$	digital angular frequency.
$f$	digital frequency such that $\omega = 2\pi f$ .
$X(\omega)$	Discrete-Time Fourier Transform of $x[n]$ defined as $X(\omega) = \sum_n x[n]e^{-j\omega n}.$
$X(\Omega)$	Continuous-Time Fourier Transform of $x(t)$ defined as $X(\Omega) = \int_{-\infty}^{\infty} x(t)e^{-j\Omega t} dt.$

**Trigonometric Identities:**

$$\begin{aligned} \sin(\alpha \pm \beta) &= \sin \alpha \cos \beta \pm \cos \alpha \sin \beta & \sin \alpha \sin \beta &= \frac{1}{2} \cos(\alpha - \beta) - \frac{1}{2} \cos(\alpha + \beta) \\ \cos(\alpha \pm \beta) &= \cos \alpha \cos \beta \mp \sin \alpha \sin \beta & \cos \alpha \cos \beta &= \frac{1}{2} \cos(\alpha + \beta) + \frac{1}{2} \cos(\alpha - \beta) \\ & & \sin \alpha \cos \beta &= \frac{1}{2} \sin(\alpha + \beta) + \frac{1}{2} \sin(\alpha - \beta) \end{aligned}$$

**Euler's Identity:**

$$e^{j\theta} = \cos \theta + j \sin \theta$$

**Discrete Fourier Transform:** The  $N$ -point DFT of a  $N$ -sample sequence  $s[n]$  is defined as:

$$S[k] = \sum_{n=0}^{N-1} s[n] W_N^{kn}, \quad k = 0, 1, \dots, N-1.$$

The sequence  $s[n]$  can be recovered from its DFT coefficients using the  $N$ -point IDFT:

$$s[n] = \frac{1}{N} \sum_{k=0}^{N-1} S[k] W_N^{-kn}, \quad n = 0, 1, \dots, N-1.$$

where  $W_N = e^{-j2\pi/N}$ . The DFT/IDFT relations can also be expressed in matrix form

$$\begin{bmatrix} S[0] \\ \vdots \\ S[N-1] \end{bmatrix} = \mathbf{W}_N \begin{bmatrix} s[0] \\ \vdots \\ s[N-1] \end{bmatrix} \quad \begin{bmatrix} s[0] \\ \vdots \\ s[N-1] \end{bmatrix} = \mathbf{W}_N^{-1} \begin{bmatrix} S[0] \\ \vdots \\ S[N-1] \end{bmatrix}$$

where the transformation matrices for  $N = 2, 3, 4$  are

$$\mathbf{W}_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$\mathbf{W}_2^{-1} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$\mathbf{W}_3 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -\frac{1}{2} - \frac{\sqrt{3}}{2}j & -\frac{1}{2} + \frac{\sqrt{3}}{2}j \\ 1 & -\frac{1}{2} + \frac{\sqrt{3}}{2}j & -\frac{1}{2} - \frac{\sqrt{3}}{2}j \end{bmatrix}$$

$$\mathbf{W}_3^{-1} = \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 \\ 1 & -\frac{1}{2} + \frac{\sqrt{3}}{2}j & -\frac{1}{2} - \frac{\sqrt{3}}{2}j \\ 1 & -\frac{1}{2} - \frac{\sqrt{3}}{2}j & -\frac{1}{2} + \frac{\sqrt{3}}{2}j \end{bmatrix}$$

$$\mathbf{W}_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}$$

$$\mathbf{W}_4^{-1} = \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & +j \end{bmatrix}$$

**Table 1:** Symmetry Properties of the Discrete-Time Fourier Transform

Sequence $x[n]$	Fourier Transform $X(e^{j\omega})$
1. $x^*[n]$	$X^*(e^{-j\omega})$
2. $x^*[-n]$	$X^*(e^{j\omega})$
3. $\mathcal{R}\{x[n]\}$	$X_e(e^{j\omega})$ (conjugate-symmetric part of $X(e^{j\omega})$ )
4. $j\mathcal{I}\{x[n]\}$	$X_o(e^{j\omega})$ (conjugate-antisymmetric part of $X(e^{j\omega})$ )
5. $x_e[n]$ (conjugate-symmetric part of $x[n]$ )	$X_R(e^{j\omega}) = \mathcal{R}\{X(e^{j\omega})\}$
6. $x_o[n]$ (conjugate-antisymmetric part of $x[n]$ )	$jX_I(e^{j\omega}) = j\mathcal{I}\{X(e^{j\omega})\}$
<i>The following properties apply only when <math>x[n]</math> is real:</i>	
7. Any real $x[n]$	$X(e^{j\omega}) = X^*(e^{-j\omega})$ (Fourier transform is conjugate symmetric)
8. Any real $x[n]$	$X_R(e^{j\omega}) = X_R(e^{-j\omega})$ (real part is even)
9. Any real $x[n]$	$X_I(e^{j\omega}) = -X_I(e^{-j\omega})$ (imaginary part is odd)
10. Any real $x[n]$	$ X(e^{j\omega})  =  X(e^{-j\omega}) $ (magnitude is even)
11. Any real $x[n]$	$\angle X(e^{j\omega}) = -\angle X(e^{-j\omega})$ (phase is odd)
12. $x_e[n]$ (even part of $x[n]$ )	$X_R(e^{j\omega})$
13. $x_o[n]$ (odd part of $x[n]$ )	$jX_I(e^{j\omega})$

**Table 2:** Discrete-Time Fourier Transform Theorems

Sequence	Fourier Transform
$x[n]$	$X(e^{j\omega})$
$y[n]$	$Y(e^{j\omega})$
1. $ax[n] + by[n]$	$aX(e^{j\omega}) + bY(e^{j\omega})$
2. $x[n - n_d]$ ( $n_d$ an integer)	$e^{-j\omega n_d} X(e^{j\omega})$
3. $e^{j\omega_0 n} x[n]$	$X(e^{j(\omega - \omega_0)})$
4. $x[-n]$	$X(e^{-j\omega})$ $X^*(e^{j\omega})$ if $x[n]$ real.
5. $nx[n]$	$j \frac{dX(e^{j\omega})}{d\omega}$
6. $x[n] * y[n]$	$X(e^{j\omega})Y(e^{j\omega})$
7. $x[n]y[n]$	$\frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\theta})Y(e^{j(\omega - \theta)})d\theta$
Parseval's theorem:	
8. $\sum_{n=-\infty}^{\infty}  x[n] ^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi}  X(e^{j\omega}) ^2 d\omega$	
9. $\sum_{n=-\infty}^{\infty} x[n]y^*[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega})Y^*(e^{j\omega})d\omega$	

**Table 3:** Discrete-Time Fourier Transform Pairs

Sequence	Fourier Transform
1. $\delta[n]$	1
2. $\delta[n - n_0]$	$e^{-j\omega n_0}$
3. 1 $(-\infty < n < \infty)$	$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega + 2\pi k)$
4. $a^n u[n]$ $( a  < 1)$	$\frac{1}{1 - ae^{-j\omega}}$
5. $u[n]$	$\frac{1}{1 - e^{-j\omega}} + \sum_{k=-\infty}^{\infty} \pi\delta(\omega + 2\pi k)$
6. $(n + 1)a^n u[n]$ $( a  < 1)$	$\frac{1}{(1 - ae^{-j\omega})^2}$
7. $\frac{r^n \sin \omega_p (n + 1)}{\sin \omega_p} u[n]$ $( r  < 1)$	$\frac{1}{1 - 2r \cos \omega_p e^{-j\omega} + r^2 e^{-j2\omega}}$
8. $\frac{\text{siñ } \omega_c n}{\pi n}$	$X(e^{j\omega}) = \begin{cases} 1, &  \omega  < \omega_c, \\ 0, & \omega_c <  \omega  \leq \pi \end{cases}$
9. $x[n] = \begin{cases} 1, & 0 \leq n \leq M \\ 0, & \text{otherwise} \end{cases}$	$\frac{\sin[\omega(M + 1)/2]}{\sin(\omega/2)} e^{-j\omega M/2}$
10. $e^{j\omega_0 n}$	$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega - \omega_0 + 2\pi k)$
11. $\cos(\omega_0 n + \phi)$	$\sum_{k=-\infty}^{\infty} [\pi e^{j\phi} \delta(\omega - \omega_0 + 2\pi k) + \pi e^{-j\phi} \delta(\omega + \omega_0 + 2\pi k)]$

**Table 4:** Some Common  $z$ -Transform Pairs

Sequence	Transform	ROC
1. $\delta[n]$	1	All $z$
2. $u[n]$	$\frac{1}{1 - z^{-1}}$	$ z  > 1$
3. $-u[-n - 1]$	$\frac{1}{1 - z^{-1}}$	$ z  < 1$
4. $\delta[n - m]$	$z^{-m}$	All $z$ except 0 (if $m > 0$ ),
5. $a^n u[n]$	$\frac{1}{1 - az^{-1}}$	$ z  >  a $
6. $-a^n u[-n - 1]$	$\frac{1}{1 - az^{-1}}$	$ z  <  a $
7. $na^n u[n]$	$\frac{az^{-1}}{(1 - az^{-1})^2}$	$ z  >  a $
8. $-na^n u[-n - 1]$	$\frac{az^{-1}}{(1 - az^{-1})^2}$	$ z  <  a $
9. $\cos(\omega_0 n)u[n]$	$\frac{1 - \cos(\omega_0)z^{-1}}{1 - 2\cos(\omega_0)z^{-1} + z^{-2}}$	$ z  > 1$
10. $\sin(\omega_0 n)u[n]$	$\frac{\sin(\omega_0)z^{-1}}{1 - 2\cos(\omega_0)z^{-1} + z^{-2}}$	$ z  > 1$
11. $r^n \cos(\omega_0 n)u[n]$	$\frac{1 - r\cos(\omega_0)z^{-1}}{1 - 2r\cos(\omega_0)z^{-1} + r^2z^{-2}}$	$ z  > r$
12. $r^n \sin(\omega_0 n)u[n]$	$\frac{r\sin(\omega_0)z^{-1}}{1 - 2r\cos(\omega_0)z^{-1} + r^2z^{-2}}$	$ z  > r$
13. $\begin{cases} a^n, & 0 \leq n \leq N - 1, \\ 0, & \text{otherwise} \end{cases}$	$\frac{1 - a^N z^{-N}}{1 - az^{-1}}$	$ z  > 0$

**Table 5:** Some  $z$ -Transform Properties

Sequence	Transform	ROC
$x[n]$	$X(z)$	$R_x$
$x_1[n]$	$X_1(z)$	$R_{x_1}$
$x_2[n]$	$X_2(z)$	$R_{x_2}$
$ax_1[n] + bx_2[n]$	$aX_1(z) + bX_2(z)$	Contains $R_{x_1} \cap R_{x_2}$
$x[n - n_0]$	$z^{-n_0} X(z)$	$R_x$ , except for the possible addition or deletion of the origin or $\infty$
$z_0^n x[n]$	$X(z/z_0)$	$ z_0  R_x$
$nx[n]$	$-\tau \frac{dX(z)}{dz}$	$R_x$
$x^*[n]$	$X^*(z^*)$	$R_x$
$\text{Re}\{x[n]\}$	$\frac{1}{2}[X(z) + X^*(z^*)]$	Contains $R_x$
$\text{Im}\{x[n]\}$	$\frac{1}{2j}[X(z) - X^*(z^*)]$	Contains $R_x$
$x^*[-n]$	$X^*(1/z^*)$	$1/R_x$
$x_1[n] * x_2[n]$	$X_1(z)X_2(z)$	Contains $R_{x_1} \cap R_{x_2}$

## Marking Scheme

**Question 1:** (a) 10, (b) 10 marks.

**Question 2:** (a) 3, (b) 4, (c) 5, (d) 4, (e) 4 marks.

**Question 3:** (a) 5, (b) 5, (c) 5, (d) 5 marks.

**Question 4:** (a) 2, (b) 4, (c) 4, (d) 4, (e) 6 marks.

**Question 5:** (a) 3, (b) 3, (c) 4, (d) 7, (e) 3 marks.

**Question 6:** (a) 7, (b) 8, (c) 5 marks.

**Question 7:** (a) 6, (b) 9, (c) 5 marks.